WELL-POSEDNESS AND ASYMPTOTIC BEHAVIOR FOR STOCHASTIC REACTION-DIFFUSION EQUATIONS WITH MULTIPLICATIVE POISSON NOISE

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Abstract. We establish well-posedness in the mild sense for a class of stochastic semilinear evolution equations with a polynomially growing quasi-monotone nonlinearity and multiplicative Poisson noise. We also study existence and uniqueness of invariant measures for the associated semigroup in the Markovian case. A key role is played by a new maximal inequality for stochastic convolutions in L_p spaces.

1. Introduction

The purpose of this paper is to obtain existence and uniqueness of solutions, as well as existence and uniqueness of invariant measures, for a class of semilinear stochastic partial differential equations driven by a discontinuous multiplicative noise. In particular, we consider the mild formulation of an equation of the type

$$du(t) + Au(t)\,dt + F(u(t))\,dt = \int_Z G(u(t-),z)\,\bar{\mu}(dt,dz) \tag{1}$$

on $L_2(D)$, with D a bounded domain of $\mathbb{R}^n$. Here $-A$ is the generator of a strongly continuous semigroup of contractions, F is a nonlinear function satisfying monotonicity and polynomial growth conditions, and $\bar{\mu}$ is a compensated Poisson measure. Precise assumptions on the data of the problem are given in Section 2 below. We would like to note that all results of this paper also hold if we add a stochastic term of the type $B(u(t))\,dW(t)$ to the right hand side of (1), where $W(t)$, $t \geq 0$ is a cylindrical Wiener process on $L_2(D)$ and B satisfies appropriate assumptions. For simplicity we concentrate on the jump part of the noise. Similarly, all results of the paper still hold with minimal modifications if we allow the functions F and G to depend also on time and to be random.

While several classes of semilinear stochastic PDEs driven by Wiener noise, also with rather general nonlinearity F, have been extensively studied (see e.g. [9, 11, 12] and references therein), a corresponding body of results for equations driven by jump noise seems to be missing. Let us mention, however, several notable exceptions: existence of local mild solutions for equations with locally Lipschitz nonlinearities has been established in

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[21]; stochastic PDEs with monotone nonlinearities driven by general martingales have been investigated in [17] in a variational setting, following the approach of [22]; an analytic approach yielding weak solutions (in the probabilistic sense) for equations with singular drift and additive Lévy noise has been developed in [24]. The more recent monograph [30] deals also with semilinear SPDEs with monotone nonlinearity and additive Lévy noise, and contains a well-posedness result under a set of regularity assumptions on F and the stochastic convolution. In particular, continuity with respect to stronger norms (more precisely, in spaces continuously embedded into $L_2(D)$) is assumed. We avoid such conditions, thus making our assumptions more transparent and much easier to verify.

Similarly, not many results are available about the asymptotic behavior of the solution to SPDEs with jump noise, while the literature for equations with continuous noise is quite rich (see the references mentioned above). In this work we show that under a suitably strong monotonicity assumption one obtains existence, uniqueness, and ergodicity of invariant measures, while a weaker monotonicity assumption is enough to obtain the existence of invariant measures.

Our main contributions could be summarized as follows: we provide a) a set of sufficient conditions for well-posedness in the mild sense for SPDEs of the form (1), which to the best of our knowledge is not contained nor can be derived from existing work; b) a new concept of generalized mild solution which allows us to treat equations with a noise coefficient G satisfying only natural integrability and continuity assumptions; c) existence of invariant measures without strong dissipativity assumptions on the coefficients of (1). It is probably worth commenting a little further on the first issue: it is in general not possible to find a triple $V \subset H \subset V'$ (see e.g. [17, 22, 31] for details) such that $A + F$ is defined from V to V' and satisfies the usual continuity, accretivity and coercivity assumptions needed for the theory to work. For this reason, general semilinear SPDEs cannot be (always) treated in the variational setting. Moreover, the Nemitskii operator associated to F is in general not locally Lipschitz on $L_2(D)$, so one cannot hope to obtain global well-posedness invoking the local well-posedness results of [21], combined with a priori estimates. Finally, while the analytic approach of [24] could perhaps be adapted to our situation, it would cover only the case of additive noise, and solutions would be obtained only in the sense of the martingale problem.

The main tool employed in the existence theory is a Bichteler-Jacod-type inequality for stochastic convolutions on L_p spaces, combined with monotonicity estimates. To obtain well-posedness for equations with general noise, also of multiplicative type, we need to relax the concept of solution we work with, in analogy to the deterministic case (see [3, 6]). Finally, we prove existence of an invariant measure by an argument based on Krylov-Bogoliubov's theorem under weak dissipativity conditions. Existence and uniqueness of an invariant measure under strong dissipativity conditions is also obtained, adapting a classical method (see e.g. [13]).

The paper is organized as follows. In Section 2 all well-posedness results are stated and proved, and Section 3 contains the results on invariant measures. Finally, we prove in the Appendix an auxiliary result used in Section 2.

Let us conclude this section with a few words about notation. Generic constants will be denoted by N, and we shall use the shorthand notation $a \lesssim b$ to mean $a \leq Nb$.

If the constant N depends on a parameter p, we shall also write $N(p)$ and $a \lesssim_p b$. Given a function $f : \mathbb{R} \to \mathbb{R}$, we shall denote its associated Nemitsky operator by the same symbol. Moreover, given an integer k, we shall write f^k for the function $\xi \mapsto f(\xi)^k$. For any topological space X we shall denote its Borel σ-field by $\mathcal{B}(X)$. We shall occasionally use standard abbreviations for stochastic integrals with respect to martingales and stochastic measures, so that $H \cdot X(t) := \int_0^t H(s)\,dX(s)$ and $\phi \star \mu(t) := \int_0^t \int \phi(s,y)\,\mu(ds,dy)$ (see e.g. [20] for more details). Given two Banach spaces E and F, we shall denote the set of all functions $f : E \to F$ such that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|_F}{|x - y|_E} < \infty$$

by $\dot{C}^{0,1}(E,F)$.

2. Well-posedness

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and E denote expectation with respect to $\mathbb{P}$. All stochastic elements will be defined on this stochastic basis, unless otherwise specified. The preditable σ-field will be denoted by $\mathcal{P}$. Let $(Z, \mathcal{Z}, m)$ be a measure space, $\bar{\mu}$ a Poisson measure on $[0,T] \times Z$ with compensator Leb$\otimes m$, where Leb stands for Lebesgue measure. We shall set, for simplicity of notation, $Z_t = (0,t] \times Z$, for $t \geq 0$, and $L_p(Z_t) := L_p(Z_t, \mathrm{Leb} \otimes m)$. Let D be an open bounded subset of $\mathbb{R}^d$ with smooth boundary ∂D, and set $H = L_2(D)$. The norm and inner product in H are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively, while the norm in $L_p(D)$, $p \geq 1$, is denoted by $|\cdot|_p$. Given a Banach space E, we shall denote the set of all E-valued random variables ξ such that $\mathbb{E}|\xi|^p < \infty$ by $\mathbb{L}_p(E)$. For compactness of notation, we also set $\mathbb{L}_p := \mathbb{L}_p(L_p(D))$. Moreover, we denote the set of all adapted processes $u : [0,T] \times \Omega \to H$ such that

$$|[u]|_p := \Big(\sup_{t\leq T} \mathbb{E}|u(t)|^p\Big)^{1/p} < \infty, \qquad \|u\|_p := \Big(\mathbb{E}\sup_{t\leq T} |u(t)|^p\Big)^{1/p} < \infty$$

by $\mathcal{H}_p(T)$ and $\mathbb{H}_p(T)$, respectively. Note that $(\mathcal{H}_p(T), |[\cdot]|_p)$ and $(\mathbb{H}_p(T), \|\cdot\|_p)$ are Banach spaces. We shall also use the equivalent norms on $\mathbb{H}_p(T)$ defined by

$$\|u\|_{p,\alpha} := \Big(\mathbb{E}\sup_{t\leq T} e^{-p\alpha t}|u(t)|^p\Big)^{1/p}, \qquad \alpha > 0,$$

and we shall denote $(\mathbb{H}_p(T), \|\cdot\|_{p,\alpha})$ by $\mathbb{H}_{p,\alpha}(T)$.

2.1. Additive noise.

Let us consider the equation

$$du(t) + Au(t)\,dt + f(u(t))\,dt = \eta u(t)\,dt + \int_Z G(t,z)\,\bar{\mu}(dt,dz), \qquad u(0) = x, \tag{2}$$

where A is a linear maximal monotone operator on H; $f : \mathbb{R} \to \mathbb{R}$ is a continuous maximal monotone function satisfying the growth condition $|f(r)| \lesssim 1 + |r|^d$ for some (fixed) $d \in [1, \infty[$; $G : \Omega \times [0,T] \times Z \times D \to \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable process, such that $G(t,z) \equiv G(\omega,t,z,\cdot)$ takes values in $H = L_2(D)$. Finally, η is just a constant and the corresponding term is added for convenience (see below). We shall assume throughout the paper that the semigroup generated by $-A$ admits a unique extension to a strongly

continuous semigroup of positive contractions on $L_{2d}(D)$ and $L_{d^*}(D)$, $d^* := 2d^2$. For simplicity of notation we shall not distinguish among the realizations of A and e^{-tA} on different $L_p(D)$ spaces, if no confusion can arise.

Remark 1. Several examples of interest satisfy the assumptions on A just mentioned. For instance, A could be chosen as the realization of an elliptic operator on D of order $2m$, $m \in \mathbb{N}$, with Dirichlet boundary conditions (see e.g. [1]). The operator $-A$ can also be chosen as the generator of a sub-Markovian strongly continuous semigroup of contractions T_t on $L_2(D)$. In fact, an argument based on the Riesz-Thorin interpolation theorem shows that T_t induces a strongly continuous sub-Markovian contraction semigroup $T_t^{(p)}$ on any $L_p(D)$, $p \in [2, +\infty[$ (see e.g. [14, Lemma 1.11] for a detailed proof). The latter class of operators includes also nonlocal operators such as, for instance, fractional powers of the Laplacian, and even more general pseudodifferential operators with negative-definite symbols – see e.g. [19] for more details and examples.

Definition 2. *Let $x \in \mathbb{L}_{2d}$. We say that $u \in \mathbb{H}_2(T)$ is a mild solution of (2) if $u(t) \in L_{2d}(D)$ $\mathbb{P}$-a.s. and*

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-s)A}\big(\eta u(s) - f(u(s))\big)\, ds + \int_{Z_t} e^{-(t-s)A} G(s,z)\, \bar{\mu}(ds,dz) \qquad (3)$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, and all integrals on the right-hand side exist.

Let us denote the class of processes G as above such that

$$\mathbb{E}\int_0^T \Big[\int_Z |G(t,z)|_p^p \, m(dz) + \Big(\int_Z |G(t,z)|_p^2 \, m(dz) \Big)^{p/2} \Big]\, ds < \infty.$$

by $\mathcal{L}_p$. Setting $d^* = 2d^2$, we shall see below that a sufficient condition for the existence of the integrals appearing in (3) is that $G \in \mathcal{L}_{d^*}$. This also explains the condition imposed on the sequence $\{G_n\}$ in the next definition.

Definition 3. *Let $x \in \mathbb{L}_2$. We say that $u \in \mathbb{H}_2(T)$ is a generalized mild solution of (2) if there exist a sequence $\{x_n\} \subset \mathbb{L}_{2d}$ and a sequence $\{G_n\} \subset \mathcal{L}_{d^*}$ with $x_n \to x$ in $\mathbb{L}_2$ and $G_n \to G$ in $\mathbb{L}_2(L_2(Z_T))$, such that $u_n \to u$ in $\mathbb{H}_2(T)$, where u_n is the mild solution of (2) with x_n and G_n replacing x and G, respectively.*

In order to establish well-posedness of the stochastic equation, we need the following maximal inequalities, that are extensions to a (specific) Banach space setting of the corresponding inequalities proved for Hilbert space valued processes in [27], with a completely different proof.

Lemma 4. *Let $E = L_p(D)$, $p \in [2, \infty)$. Assume that $g : \Omega \times [0,T] \times Z \times D \to \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable function such that the expectation on the right-hand side of (4) below is finite. Then there exists a constant $N = N(p,T)$ such that*

$$\begin{aligned} \mathbb{E}\sup_{t \le T} \Big| \int_0^t \int_Z g(s,z)\, \bar{\mu}(ds,dz) \Big|_E^p \\ \le N\mathbb{E}\int_0^T \Big[\int_Z |g(s,z)|_E^p \, m(dz) + \Big(\int_Z |g(s,z)|_E^2 \, m(dz) \Big)^{p/2} \Big]\, ds, \qquad (4) \end{aligned}$$

where $(p,T) \mapsto N$ is continuous. Furthermore, let $-A$ be the generator of a strongly continuous semigroup e^{-tA} of positive contractions on E. Then one also has

$$\mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z e^{-(t-s)A}g(s,z)\,\bar\mu(ds,dz)\Big|^p \\ \le N\mathbb{E}\int_0^T\Big[\int_Z |g(s,z)|_E^p\,m(dz)+\Big(\int_Z |g(s,z)|_E^2\,m(dz)\Big)^{p/2}\Big]\,ds, \quad (5)$$

where N is the same constant as in (4).

Proof. Since the right-hand side of (4) is finite, Fubini's theorem implies that there exists $D' \subset D$, $\mathrm{Leb}(D\setminus D')=0$, such that

$$\mathbb{E}\int_0^T\Big[\int_Z |g(s,z,\xi)|^p\,m(dz)+\Big(\int_Z |g(s,z,\xi)|^2\,m(dz)\Big)^{p/2}\Big]\,ds<\infty$$

for all $\xi \in D'$. Fix $\xi\in D'$. By the Bichteler-Jacod inequality for real-valued integrands (see e.g. [4, 27]) we have

$$\mathbb{E}\Big|\int_0^T\!\!\int_Z g(s,z,\xi)\,\bar\mu(ds,dz)\Big|^p \\ \lesssim_{p,T}\mathbb{E}\int_0^T\Big[\int_Z |g(s,z,\xi)|^p\,m(dz)+\Big(\int_Z |g(s,z,\xi)|^2\,m(dz)\Big)^{p/2}\Big]\,ds. \quad (6)$$

Furthermore, Fubini's theorem for integrals with respect to random measures (see e.g. [23] or [5, App. A]) yields

$$\mathbb{E}\Big|\int_0^T\!\!\int_Z g(s,z)\,\bar\mu(ds,dz)\Big|_E^p=\int_D\mathbb{E}\Big|\int_0^T\!\!\int_Z g(s,z,\xi)\,\bar\mu(ds,dz)\Big|^p\,d\xi,$$

hence also

$$\mathbb{E}\Big|\int_0^T\!\!\int_Z g(s,z)\,\bar\mu(ds,dz)\Big|_E^p\lesssim_{p,T}\mathbb{E}\int_0^T\!\!\int_Z\!\int_D |g(s,z,\xi)|^p\,d\xi\,m(dz)\,ds \\ +\mathbb{E}\int_0^T\!\!\int_D\Big(\int_Z |g(s,z,\xi)|^2\,m(dz)\Big)^{p/2}\,d\xi\,ds.$$

However, Minkowski's inequality (see e.g. [25, Thm. 2.4]) implies that the second term on the right-hand side of the previous inequality is less or equal to

$$\mathbb{E}\int_0^T\Big(\int_Z\Big(\int_D |g(s,z,\xi)|^p\,d\xi\Big)^{2/p}m(dz)\Big)^{p/2}ds=\mathbb{E}\int_0^T\Big(\int_Z |g(s,z)|_E^2\,m(dz)\Big)^{p/2}ds.$$

We have thus proved that

$$\mathbb{E}|g\star\bar\mu(T)|_E^p\lesssim_{p,T}\mathbb{E}\int_0^T\Big[\int_Z |g(s,z)|_E^p\,m(dz)+\Big(\int_Z |g(s,z)|_E^2\,m(dz)\Big)^{p/2}\Big]\,ds.$$

Estimate (4) now follows immediately, by Doob's inequality, provided we can prove that $g\star\bar\mu$ is an E-valued martingale. For this it suffices to prove that

$$\mathbb{E}\big[\langle g\star\bar\mu(t)-g\star\bar\mu(s),\phi\rangle\big|\mathcal{F}_s\big]=0, \qquad 0\le s\le t\le T,$$

for all $\phi \in C_c^\infty(D)$, the space of infinitely differentiable functions with compact support on D. In fact, we have, by the stochastic Fubini theorem,

$$\begin{aligned}\langle g \star \bar\mu(t) - g \star \bar\mu(s), \phi\rangle &= \Big\langle \int_{(s,t]}\int_Z g(r,z)\,\bar\mu(dr,dz), \phi\Big\rangle \\ &= \int_{(s,t]}\int_Z\int_D g(r,z,\xi)\phi(\xi)\,d\xi\,\bar\mu(dr,dz),\end{aligned}$$

where the last term has $\mathcal{F}_s$-conditional expectation equal to zero by well-known properties of Poisson measures. In order for the above computation to be rigorous, we need to show that the last stochastic integral is well defined: using Hölder's inequality and recalling that $g \in \mathcal{L}_p$, we get

$$\begin{aligned}&\mathbb{E}\int_{(s,t]}\int_Z \Big[\int_D g(r,z,\xi)\phi(\xi)\,d\xi\Big]^2 m(dz)\,dr \le |\phi|^2_{\frac{p}{p-1}}\mathbb{E}\int_0^T\int_Z |g(s,z)|_E^2\,m(dz)\,ds \\ &\quad\le |\phi|^2_{\frac{p}{p-1}} T^{p/(p-2)}\Big(\mathbb{E}\int_0^T\Big(\int_Z |g(s,z)|_p^2\,m(dz)\Big)^{p/2} ds\Big)^{2/p} < \infty.\end{aligned}$$

In order to extend the result to stochastic convolutions, we need a dilation theorem due to Fendler [15, Thm. 1]. In particular, there exist a measure space $(Y,\mathcal{A},n)$, a strongly continuous group of isometries $T(t)$ on $\bar E := L_p(Y,n)$, an isometric linear embedding $j : L_p(D) \to L_p(Y,n)$, and a contractive projection $\pi : L_p(Y,n) \to L_p(D)$ such that $j \circ e^{tA} = \pi \circ T(t) \circ j$ for all $t \ge 0$. Then we have, recalling that the operator norms of π and $T(t)$ are less than or equal to one,

$$\begin{aligned}&\mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z e^{-(t-s)A}g(s,z)\,\bar\mu(ds,dz)\Big|_E^p \\ &\quad= \mathbb{E}\sup_{t\le T}\Big|\pi T(t)\int_0^t\int_Z T(-s)j(g(s,z))\,\bar\mu(ds,dz)\Big|_{\bar E}^p \\ &\quad\le |\pi|^p\sup_{t\le T}|T(t)|^p\,\mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z T(-s)j(g(s,z))\,\bar\mu(ds,dz)\Big|_{\bar E}^p \\ &\quad\le \mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z T(-s)j(g(s,z))\,\bar\mu(ds,dz)\Big|_{\bar E}^p\end{aligned}$$

Now inequality (4) implies that there exists a constant $N = N(p,T)$ such that

$$\begin{aligned}&\mathbb{E}\sup_{t\le T}\Big|\int_0^t\int_Z e^{-(t-s)A}g(s,z)\,\bar\mu(ds,dz)\Big|_E^p \\ &\quad\le N\mathbb{E}\int_0^T\Big[\int_Z |T(-s)j(g(s,z))|_{\bar E}^p\,m(dz) + \Big(\int_Z |T(-s)j(g(s,z))|_{\bar E}^2\,m(dz)\Big)^{p/2}\Big]ds \\ &\quad\le N\mathbb{E}\int_0^T\Big[\int_Z |g(s,z)|_E^p\,m(dz) + \Big(\int_Z |g(s,z)|_E^2\,m(dz)\Big)^{p/2}\Big]ds\end{aligned}$$

where we have used again that $T(t)$ is a unitary group and that the norms of $\bar E$ and E are equal. □

Remark 5. (i) Inequality (4) could have also been proved following the same strategy of [27], using a suitable version of Itô's formula for Banach-space valued processes (see e.g. [16]).
(ii) The idea of using dilation theorems to extend results from stochastic integrals to stochastic convolutions has been introduced, to the best of our knowledge, in [18].
(iii) Since $g \star \bar{\mu}$ is a martingale taking values in $L_p(D)$, it has a càdlàg modification, as it follows by a theorem of Brooks and Dinculeanu (see [8, Thm. 3]). Moreover, the stochastic convolution also admits a càdlàg modification by the dilation method, as in [18] or [30, p. 161].

We shall need to regularize the monotone nonlinearity f by its Yosida approximation f_λ, $\lambda > 0$. In particular, let $J_\lambda(x) = (I + \lambda f)^{-1}(x)$, $f_\lambda(x) = \lambda^{-1}(x - J_\lambda(x))$. It is well known that $f_\lambda(x) = f(J_\lambda(x))$ and $f_\lambda \in \dot{C}^{0,1}(\mathbb{R})$ with Lipschitz constant bounded by $2/\lambda$. For more details on maximal monotone operators and their approximations see e.g. [2, 6]. Let us consider the regularized equation

$$du(t) + Au(t)\,dt + f_\lambda(u(t))\,dt = \eta u(t)\,dt + \int_Z G(t,z)\,\bar{\mu}(dt,dz), \qquad u(0) = x, \tag{7}$$

which admits a unique càdlàg mild solution $u_\lambda \in \mathbb{H}_2(T)$ because $-A$ is the generator of a strongly continuous semigroup of contractions and f_λ is Lipschitz (see e.g. [21, 27, 30]).

We shall now establish an a priori estimate for solutions of the regularized equations.

Lemma 6. *Assume that $x \in \mathbb{L}_{2d}$ and $G \in \mathcal{L}_{d^*}$. Then there exists a constant $N = N(T, d, \eta, |D|)$ such that*

$$\mathbb{E}\sup_{t \le T} |u_\lambda(t)|_{2d}^{2d} \le N\big(1 + \mathbb{E}|x|_{2d}^{2d}\big). \tag{8}$$

Proof. We proceed by the technique of "subtracting the stochastic convolution": set

$$y_\lambda(t) = u_\lambda(t) - \int_0^t e^{-(t-s)A} G(s,z)\,\bar{\mu}(ds,dz) =: u_\lambda(t) - G_A(t), \qquad t \le T,$$

where

$$G_A(t) := \int_0^t\!\!\int_Z e^{-(t-s)A} G(s,z)\,\bar{\mu}(ds,dz).$$

Then y_λ is also a mild solution in $L_2(D)$ of the deterministic equation with random coefficients

$$y_\lambda'(t) + Ay_\lambda(t) + f_\lambda(y_\lambda(t) + G_A(t)) = \eta y_\lambda(t) + \eta G_A(t), \qquad y_\lambda(0) = x, \tag{9}$$

$\mathbb{P}$-a.s., where $\phi'(t) := d\phi(t)/dt$. We are now going to prove that y_λ is also a mild solution of (9) in $L_{2d}(D)$. Setting

$$\tilde{f}_\lambda(t,y) := f_\lambda(y + G_A(t)) - \eta(y + G_A(t))$$

and rewriting (9) as

$$y_\lambda'(t) + Ay_\lambda(t) + \tilde{f}_\lambda(t, y_\lambda(t)) = 0,$$

we conclude that (9) admits a unique mild solution in $L_{2d}(D)$ by Proposition 16 below (see the Appendix).

Let $y_{\lambda\beta}$ be the strong solution in $L_{2d}(D)$ of the equation

$$y_{\lambda\beta}'(t) + A_\beta y_{\lambda\beta}(t) + f_\lambda(y_{\lambda\beta}(t) + G_A(t)) = \eta y_{\lambda\beta}(t) + \eta G_A(t), \qquad y_\lambda(0) = x, \tag{10}$$

which exists and is unique because the Yosida approximation A_β is a bounded operator on $L_{2d}(D)$. Let us recall that the duality map $J : L_{2d}(D) \to L_{\frac{2d}{2d-1}}(D)$ is single valued and defined by

$$J(\phi) : \xi \mapsto |\phi(\xi)|^{2d-2}\phi(\xi)|\phi|_{2d}^{2-2d}$$

for almost all $\xi \in D$. Moreover, since $L_{\frac{2d}{2d-1}}(D)$ is uniformly convex, $J(\phi)$ coincides with the Gâteaux derivative of $\phi \mapsto |\phi|_{2d}^2/2$. Therefore, multiplying (in the sense of the duality product of $L_{2d}(D)$ and $L_{\frac{2d}{2d-1}}(D)$) both sides of (10) by the function $J(y_{\lambda\beta}(t))|y_{\lambda\beta}(t)|_{2d}^{2d-2} = y_{\lambda\beta}(t)^{2d-1}$, we get

$$\begin{aligned}\frac{1}{2d}\frac{d}{dt}|y_{\lambda\beta}(t)|_{2d}^{2d} &+ \langle A_\beta y_{\lambda\beta}(t), J(y_{\lambda\beta}(t))\rangle |y_{\lambda\beta}(t)|_{2d}^{2d-2} \\ &+ \langle f_\lambda(y_{\lambda\beta}(t) + G_A(t)), y_{\lambda\beta}(t)^{2d-1}\rangle = \eta|y_{\lambda\beta}(t)|_{2d}^{2d} + \eta\langle y_{\lambda\beta}(t)^{2d-1}, G_A(t)\rangle.\end{aligned}$$

Since A is m-accretive in $L_{2d}(D)$ (more precisely, A is an m-accretive subset of $L_{2d}(D)\times L_{2d}(D)$), its Yosida approximation $A_\beta = A(I+\beta A)^{-1}$ is also m-accretive (see e.g. [2, Prop. 2.3.2]), thus the second term on the left hand side is positive because J is single-valued. Moreover, we have, omitting the dependence on t for simplicity of notation,

$$\begin{aligned}f_\lambda(y_{\lambda\beta} + G_A)y_{\lambda\beta}^{2d-1} &= \big(f_\lambda(y_{\lambda\beta} + G_A) - f_\lambda(G_A)\big)y_{\lambda\beta}y_{\lambda\beta}^{2d-2} + f_\lambda(G_A)y_{\lambda\beta}^{2d-1} \\ &\geq f_\lambda(G_A)y_{\lambda\beta}^{2d-1} \qquad (t,\xi)\text{-a.e.},\end{aligned}$$

as it follows by the monotonicity of f_λ. Therefore we can write

$$\begin{aligned}\frac{1}{2d}\ \frac{d}{dt}|y_{\lambda\beta}(t)|_{2d}^{2d} &\leq \eta|y_{\lambda\beta}(t)|_{2d}^{2d} + \langle \eta G_A(t) - f_\lambda(G_A(t)), y_{\lambda\beta}(t)^{2d-1}\rangle \\ &\leq \eta|y_{\lambda\beta}(t)|_{2d}^{2d} + |\eta G_A(t) - f_\lambda(G_A)|_{2d}\ |y_{\lambda\beta}(t)^{2d-1}|_{\frac{2d}{2d-1}} \\ &= \eta|y_{\lambda\beta}(t)|_{2d}^{2d} + |\eta G_A(t) - f_\lambda(G_A)|_{2d}\ |y_{\lambda\beta}(t)|_{2d}^{2d-1} \\ &\leq \eta|y_{\lambda\beta}(t)|_{2d}^{2d} + \frac{1}{2d}|\eta G_A(t) - f_\lambda(G_A)|_{2d}^{2d} + \frac{2d-1}{2d}|y_{\lambda\beta}(t)|_{2d}^{2d},\end{aligned}$$

where we have used Hölder's and Young's inequalities with conjugate exponents $2d$ and $2d/(2d-1)$. A simple computation reveals immediately that there exists a constant N depending only on d and η such that

$$|\eta G_A(t) - f_\lambda(G_A)|_{2d}^{2d} \leq N(1 + |G_A(t)|_{2d^2}^{2d^2}).$$

We thus arrive at the inequality

$$\frac{1}{2d}\frac{d}{dt}|y_{\lambda\beta}(t)|_{2d}^{2d} \leq \big(\eta + \frac{2d-1}{2d}\big)|y_{\lambda\beta}(t)|_{2d}^{2d} + N\big(1 + |G_A(t)|_{2d^2}^{2d^2}\big),$$

and Gronwall's inequality yields

$$|y_{\lambda\beta}(t)|_{2d}^{2d} \lesssim_{d,\eta} 1 + |x|_{2d}^{2d} + |G_A(t)|_{2d^2}^{2d^2},$$

hence also, thanks to (5) and the hypothesis that $G \in \mathcal{L}_{d^*}$,

$$\mathbb{E}\sup_{t\leq T}|y_{\lambda\beta}(t)|_{2d}^{2d} \leq N(1 + \mathbb{E}|x|_{2d}^{2d}).$$

where the constant N does not depend on λ. Let us now prove that $y_{\lambda\beta} \to y_\lambda$ in $\mathbb{H}_2(T)$ as $\beta \to 0$: we have

$$\begin{aligned}
|y_{\lambda\beta}(t) - y_\lambda(t)| &\leq |(e^{-tA_\beta} - e^{-tA})y_\lambda(0)| \\
&\quad + \int_0^t \left| e^{-(t-s)A_\beta}\tilde{f}_\lambda(s, y_{\lambda\beta}(s)) - e^{-(t-s)A}\tilde{f}_\lambda(s, y_\lambda(s))\right| ds \\
&\leq |(e^{-tA_\beta} - e^{-tA})y_\lambda(0)| \\
&\quad + \int_0^t |e^{-(t-s)A_\beta} - e^{-(t-s)A}|\, |\tilde{f}_\lambda(s, y_\lambda(s))|\, ds \\
&\quad + \int_0^t |e^{-(t-s)A_\beta}|\, |\tilde{f}_\lambda(s, y_{\lambda\beta}(s)) - \tilde{f}_\lambda(s, y_\lambda(s))|\, ds.
\end{aligned}$$

Setting $a(\beta) := \sup_{t\leq T} |e^{-tA_\beta} - e^{-tA}|$ and recalling that f_λ has Lipschitz constant bounded by $2/\lambda$, we obtain

$$\begin{aligned}
\mathbb{E}\sup_{t\leq T} |y_{\lambda\beta}(t) - y_\lambda(t)|^2 &\lesssim_T a(\beta)|y_\lambda(0)| + a(\beta)\mathbb{E}\sup_{t\leq T} |\tilde{f}_\lambda(s, y(s))|^2 \\
&\quad + M_T(2/\lambda + \eta) \int_0^T \mathbb{E}\sup_{s\leq t} |y_{\lambda\beta}(s) - y_\lambda(s)|^2\, dt,
\end{aligned}$$

where the second term on the right-hand side is finite again because f_λ is Lipschitz and y_λ, G_A belong to $\mathbb{H}_2(T)$. Since $a(\beta) \to 0$ as $\beta \to 0$ by well-known properties of Yosida approximations, Gronwall's lemma implies the claim. Therefore, by a lower semicontinuity argument, we get

$$\mathbb{E}\sup_{t\leq T} |y_\lambda(t)|_{2d}^{2d} \leq N(1 + \mathbb{E}|x|_{2d}^{2d}).$$

By definition of y_λ we also infer that

$$\mathbb{E}\sup_{t\leq T} |u_\lambda(t)|_{2d}^{2d} \lesssim_d \mathbb{E}\sup_{t\leq T} |y_\lambda(t)|_{2d}^{2d} + \mathbb{E}\sup_{t\leq T} |G_A(t)|_{2d}^{2d}.$$

Since

$$\mathbb{E}\sup_{t\leq T} |G_A(t)|_{2d}^{2d} \lesssim_{|D|} \mathbb{E}\sup_{t\leq T} |G_A(t)|_{2d^2}^{2d} \lesssim 1 + \mathbb{E}\sup_{t\leq T} |G_A(t)|_{2d^2}^{2d^2},$$

we conclude

$$\mathbb{E}\sup_{t\leq T} |u_\lambda(t)|_{2d}^{2d} \lesssim_{T,d,\eta,|D|} 1 + \mathbb{E}|x|_{2d}^{2d}. \qquad \square$$

The a priori estimate just obtained for the solution of the regularized equation allows us to construct a mild solution of the original equation as a limit in $\mathbb{H}_2(T)$, as the following proposition shows.

Proposition 7. *Assume that $x \in \mathbb{L}_{2d}$ and $G \in \mathcal{L}_{d^*}$. Then equation (2) admits a unique càdlàg mild solution in $\mathbb{H}_2(T)$ which satisfies the estimate*

$$\mathbb{E}\sup_{t\leq T} |u(t)|_{2d}^{2d} \leq N(1 + \mathbb{E}|x|_{2d}^{2d})$$

with $N = N(T, d, \eta, |D|)$. Moreover, we have $x \mapsto u(x) \in \dot{C}^{0,1}(\mathbb{L}_2, \mathbb{H}_2(T))$.

Proof. Let u_λ be the solution of the regularized equation (7), and $u_{\lambda\beta}$ be the strong solution of (7) with A replaced by A_β studied in the proof of Lemma 6 (or see [28, Thm. 34.7]). Then $u_{\lambda\beta} - u_{\mu\beta}$ solves $\mathbb{P}$-a.s. the equation

$$\frac{d}{dt}(u_{\lambda\beta}(t) - u_{\mu\beta}(t)) + A_\beta(u_{\lambda\beta}(t) - u_{\mu\beta}(t)) \\ + f_\lambda(u_{\lambda\beta}(t)) - f_\mu(u_{\mu\beta}(t)) = \eta(u_{\lambda\beta}(t) - u_{\mu\beta}(t)). \quad (11)$$

Note that we have

$$\begin{aligned} u_{\lambda\beta} - u_{\mu\beta} &= u_{\lambda\beta} - J_\lambda u_{\lambda\beta} + J_\lambda u_{\lambda\beta} - J_\mu u_{\mu\beta} + J_\mu u_{\mu\beta} - u_{\mu\beta} \\ &= \lambda f_\lambda(u_{\lambda\beta}) + J_\lambda u_{\lambda\beta} - J_\mu u_{\mu\beta} - \mu f_\mu(u_{\mu\beta}), \end{aligned}$$

hence, recalling that $f_\lambda(u_{\lambda\beta}) = f(J_\lambda u_{\lambda\beta})$,

$$\begin{aligned} \langle f_\lambda(u_{\lambda\beta}) - f_\mu(u_{\mu\beta}), u_{\lambda\beta} - u_{\mu\beta}\rangle &\geq \langle f_\lambda(u_{\lambda\beta}) - f_\mu(u_{\mu\beta}), \lambda f_\lambda(u_{\lambda\beta}) - \mu f_\mu(u_{\mu\beta})\rangle \\ &\geq \lambda|f_\lambda(u_{\lambda\beta})|^2 + \mu|f_\mu(u_{\mu\beta})|^2 - (\lambda+\mu)|f_\lambda(u_{\lambda\beta})||f_\mu(u_{\mu\beta})| \\ &\geq -\frac{\mu}{2}|f_\lambda(u_{\lambda\beta})|^2 - \frac{\lambda}{2}|f_\mu(u_{\mu\beta})|^2, \end{aligned}$$

thus also, by the monotonicity of A,

$$\frac{d}{dt}|u_{\lambda\beta}(t) - u_{\mu\beta}(t)|^2 - 2\eta|u_{\lambda\beta}(t) - u_{\mu\beta}(t)|^2 \leq \mu|f_\lambda(u_{\lambda\beta}(t))|^2 + \lambda|f_\mu(u_{\mu\beta}(t))|^2.$$

Multiplying both sides by $e^{-2\eta t}$ and integrating we get

$$e^{-2\eta t}|u_{\lambda\beta}(t) - u_{\mu\beta}(t)|^2 \leq \int_0^t e^{-2\eta s}\big(\mu|f_\lambda(u_{\lambda\beta}(s))|^2 + \lambda|f_\mu(u_{\mu\beta}(s))|^2\big)\,ds.$$

Since $u_{\lambda\beta} \to u_\lambda$ in $\mathbb{H}_2(T)$ as $\beta \to 0$ (as shown in the proof of Lemma 6) and f_λ is Lipschitz, we can pass to the limit as $\beta \to 0$ in the previous equation, which then holds with $u_{\lambda\beta}$ and $u_{\mu\beta}$ replaced by u_λ and u_μ, respectively. Taking supremum and expectation we thus arrive at

$$\mathbb{E}\sup_{t\leq T}|u_\lambda(t) - u_\mu(t)|^2 \leq e^{2\eta T}T(\lambda+\mu)\mathbb{E}\sup_{t\leq T}\big(|f_\lambda(u_\lambda(t))|^2 + |f_\mu(u_\mu(t))|^2\big).$$

Recalling that $|f_\lambda(x)| \leq |f(x)|$ for all $x \in \mathbb{R}$, Lemma 6 yields

$$\mathbb{E}\sup_{t\leq T}|f_\lambda(u_\lambda(t))|^2 \leq \mathbb{E}\sup_{t\leq T}|f(u_\lambda(t))|^2 \lesssim \mathbb{E}\sup_{t\leq T}|u_\lambda(t)|_{2d}^{2d} \leq N(1+\mathbb{E}|x|_{2d}^{2d}), \quad (12)$$

where the constant N does not depend on λ, hence

$$\mathbb{E}\sup_{t\leq T}|u_\lambda(t) - u_\mu(t)|^2 \lesssim_T (\lambda+\mu)\big(1+\mathbb{E}|x|_{2d}^{2d}\big),$$

which shows that $\{u_\lambda\}$ is a Cauchy sequence in $\mathbb{H}_2(T)$, and in particular there exists $u \in \mathbb{H}_2(T)$ such that $u_\lambda \to u$ in $\mathbb{H}_2(T)$. Moreover, since u_λ is càdlàg and the subset of càdlàg processes in $\mathbb{H}_2(T)$ is closed, we infer that u is itself càdlàg.

Recalling that $f_\lambda(x) = f(J_\lambda(x))$, $J_\lambda x \to x$ as $\lambda \to 0$, thanks to the dominated convergence theorem and (12) we can pass to the limit as $\lambda \to 0$ in the equation

$$u_\lambda(t) = e^{-tA}x - \int_0^t e^{-(t-s)A}f_\lambda(u_\lambda(s))\,ds + \eta\int_0^t e^{-(t-s)A}u_\lambda(s)\,ds + G_A(t),$$

thus showing that u is a mild solution of (2).

The estimate for $\mathbb{E}\sup_{t\leq T}|u(t)|_{2d}^{2d}$ is an immediate consequence of (8).

We shall now prove uniqueness. In order to simplify notation a little, we shall assume that f is η-accretive, i.e. that $r \mapsto f(r)+\eta r$ is accretive, and consequently we shall drop the first term on the right hand side of (2). This is of course completely equivalent to the original setting. Let $\{e_k\}_{k\in\mathbb{N}} \subset D(A^*)$ be an orthonormal basis of H and $\varepsilon > 0$. Denoting two solutions of (2) by u and v, we have

$$\begin{aligned}\big\langle (I+\varepsilon A^*)^{-1}e_k, u(t)-v(t)\big\rangle = &-\int_0^t \big\langle A^*(I+\varepsilon A^*)^{-1}e_k, u(s)-v(s)\big\rangle\, ds\\ &-\int_0^t \big\langle (I+\varepsilon A^*)^{-1}e_k, f(u(s))-f(v(s))\big\rangle\, ds\end{aligned}$$

for all $k\in\mathbb{N}$. Therefore, by Itô's formula,

$$\begin{aligned}&\big\langle (I+\varepsilon A^*)^{-1}e_k, u(t)-v(t)\big\rangle^2\\ &\quad= -2\int_0^t \big\langle A^*(I+\varepsilon A^*)^{-1}e_k, u(s)-v(s)\big\rangle \big\langle (I+\varepsilon A^*)^{-1}e_k, u(s)-v(s)\big\rangle\, ds\\ &\qquad -2\int_0^t \big\langle (I+\varepsilon A^*)^{-1}e_k, f(u(s))-f(v(s))\big\rangle \big\langle (I+\varepsilon A^*)^{-1}e_k, u(s)-v(s)\big\rangle\, ds.\end{aligned}$$

Summing over k and recalling that $(I+\varepsilon A^*)^{-1} = \big((I+\varepsilon A)^{-1}\big)^*$, we obtain

$$\begin{aligned}&\big|(I+\varepsilon A)^{-1}(u(t)-v(t))\big|^2\\ &\quad= -2\int_0^t \big\langle A(I+\varepsilon A)^{-1}(u(s)-v(s)), (I+\varepsilon A)^{-1}(u(s)-v(s))\big\rangle\, ds\\ &\qquad -2\int_0^t \big\langle (I+\varepsilon A)^{-1}(f(u(s))-f(v(s))), (I+\varepsilon A)^{-1}(u(s)-v(s))\big\rangle\, ds.\end{aligned}$$

Using the monotonicity of A and then letting ε tend to zero and we are left with

$$\begin{aligned}|u(t)-v(t)|^2 &\leq -2\int_0^t \langle f(u(s))-f(v(s)), u(s)-v(s)\rangle\, ds\\ &\leq 2\eta\int_0^t |u(s)-v(s)|^2\, ds,\end{aligned}$$

which immediately implies that $u=v$ by Gronwall's inequality.

Let us now prove Lipschitz continuity of the solution map. Set $u^1 := u(x_1)$, $u^2 := u(x_2)$, and denote the strong solution of (2) with A replaced by A_β, f replaced by f_λ, and initial condition x_i, $i=1,2$, by $u^i_{\lambda\beta}$, $i=1,2$, respectively. Then we have, omitting the dependence on time for simplicity,

$$(u^1_{\lambda\beta}-u^2_{\lambda\beta})' + A_\beta(u^1_{\lambda\beta}-u^2_{\lambda\beta}) + f_\lambda(u^1_{\lambda\beta}) - f_\lambda(u^2_{\lambda\beta}) = \eta(u^1_{\lambda\beta}-u^2_{\lambda\beta})$$

$\mathbb{P}$-a.s. in the strong sense. Multiplying, in the sense of the scalar product of $L_2(D)$, both sides by $u^1_{\lambda\beta}-u^2_{\lambda\beta}$ and taking into account the monotonicity of A and f, we get

$$\frac{1}{2}|u^1_{\lambda\beta}(t)-u^2_{\lambda\beta}(t)|^2 \leq |x_1-x_2|^2 + \eta\int_0^t |u^1_{\lambda\beta}(s)-u^2_{\lambda\beta}(s)|^2\, ds,$$

which implies, by Gronwall's inequality and obvious estimates,

$$\mathbb{E}\sup_{t\leq T}|u^1_{\lambda\beta}(t)-u^2_{\lambda\beta}(t)|^2 \leq e^{2\eta T}\mathbb{E}|x_1-x_2|^2.$$

Since, as seen above, $u^i_{\lambda\beta}\to u^i$, $i=1,2$, in $\mathbb{H}_2(T)$ as $\beta\to 0$, $\lambda\to 0$, we conclude by the dominated convergence theorem that $\|u^1-u^2\|_2\leq e^{\eta T}|x_1-x_2|_{\mathbb{L}_2}$. □

Remark 8. We would like to emphasize that proving uniqueness treating mild solutions as if they were strong solutions, as is very often done in the literature, does not appear to have a clear justification, unless the nonlinearity is Lipschitz continuous. In fact, if u is a mild solution of (2) and u_β is a mild (or even strong) solution of the equation obtained by replacing A with A_β in (2), one would at least need to know that u_β converges to the given solution u, which is not clear at all and essentially equivalent to what one wants to prove, namely uniqueness.

In order to establish well-posedness in the generalized mild sense, we need the following a priori estimates, which are based on Itô's formula for the square of the norm and regularizations.

Lemma 9. *Let x_1, $x_2\in\mathbb{L}_{2d}$, G_1, $G_2\in\mathcal{L}_{d^*}$, and u^1, u^2 be mild solutions of (2) with $x=x_1$, $G=G_1$ and $x=x_2$, $G=G_2$, respectively. Then one has*

$$e^{-2\eta t}\mathbb{E}|u^1(t)-u^2(t)|^2\leq \mathbb{E}|x_1-x_2|^2+\mathbb{E}\int_{Z_t}|G_1(s,z)-G_2(s,z)|^2\,m(dz)\,ds \qquad (13)$$

and

$$\mathbb{E}\sup_{t\leq T}|u^1(t)-u^2(t)|^2\lesssim_T \mathbb{E}|x_1-x_2|^2+\mathbb{E}\int_{Z_T}|G_1(s,z)-G_2(s,z)|^2\,m(dz)\,ds. \qquad (14)$$

Proof. Let u_λ and $u_{\lambda\beta}$ be defined as in the proof of Proposition 7. Set $w^i(t)=e^{-\eta t}u^i_{\lambda\beta}(t)$. Itô's formula for the square of the norm in H yields

$$|w^1(t)-w^2(t)|^2=2\int_0^t\langle w^1(s-)-w^2(s-),dw^1(s)-dw^2(s)\rangle+[w^1-w^2](t),$$

i.e.

$$\begin{aligned}
e^{-2\eta t}|u^1_{\lambda\beta}(t)-u^2_{\lambda\beta}(t)|^2 &+2\int_0^t e^{-2\eta s}\langle A_\beta(u^1_{\lambda\beta}(s)-u^2_{\lambda\beta}(s)),u^1_{\lambda\beta}(s)-u^2_{\lambda\beta}(s)\rangle\,ds\\
&+2\int_0^t e^{-2\eta s}\langle f_\lambda(u^1_{\lambda\beta}(s))-f_\lambda(u^2_{\lambda\beta}(s)),u^1_{\lambda\beta}(s)-u^2_{\lambda\beta}(s)\rangle\,ds\\
&\leq |x_1-x_2|^2+[w^1-w^2](t)+M(t),
\end{aligned}$$

where M is a local martingale. In particular, since A and f are monotone, we are left with

$$e^{-2\eta t}|u^1_{\lambda\beta}(t)-u^2_{\lambda\beta}(t)|^2\leq|x_1-x_2|^2+[w^1-w^2](t)+M(t). \qquad (15)$$

In particular, taking expectations on both sides (along a suitable localizing sequence of stopping times if necessary), we obtain

$$e^{-2\eta t}\mathbb{E}|u^1_{\lambda\beta}(t)-u^2_{\lambda\beta}(t)|^2\leq\mathbb{E}|x_1-x_2|^2+\mathbb{E}\int_{Z_t}|G_1(s,z)-G_2(s,z)|^2\,m(dz)\,ds,$$

where we have used the identity

$$\mathbb{E}[w^1 - w^2](t) = \mathbb{E}\langle w^1 - w^2\rangle(t) = \mathbb{E}\int_{Z_t} e^{-2\eta s}|G_1(s,z) - G_2(s,z)|^2\, m(dz)\, ds. \tag{16}$$

Recalling that $u^i_{\lambda\beta} \to u^i_\lambda$, $i = 1, 2$, in $\mathbb{H}_2(T)$ as β go to zero (see the proof of Lemma 6 or e.g. [27, Prop. 3.11]), we get that the above estimate holds true for u^1_λ, u^2_λ replacing $u^1_{\lambda\beta}$, $u^2_{\lambda\beta}$, respectively. Finally, since mild solutions are obtained as limits in $\mathbb{H}_2(T)$ of regularized solutions for $\lambda \to 0$, (13) follows.

By (15) and (16) we get

$$\begin{aligned}\mathbb{E}\sup_{t\leq T} e^{-2\eta t}|u^1_{\lambda\beta}(t) - u^2_{\lambda\beta}(t)|^2 \leq \mathbb{E}|x_1 - x_2|^2 + \mathbb{E}\int_{Z_T} |G_1(s,z) - G_2(s,z)|^2\, m(dz)\, ds \\ + \mathbb{E}\sup_{t\leq T}|M(t)|.\end{aligned}$$

Note that

$$M(t) = 2\int_{Z_t}\big\langle w^1_- - w^2_-, (G_1(s,z) - G_2(s,z))\,\bar\mu(ds,dz)\big\rangle = 2(w^1_- - w^2_-)\cdot(X^1 - X^2),$$

where $X^i := G_i * \bar\mu$, $i = 1, 2$. Thanks to Davis' and Young's inequalities we can write

$$\begin{aligned}\mathbb{E}\sup_{t\leq T}|M(t)| &\leq 6\mathbb{E}[(w^1_- - w^2_-)\cdot(X^1 - X^2)](T)^{1/2} \\ &\leq 6\mathbb{E}\big(\sup_{t\leq T}|w^1(t) - w^2(t)|\big)[X^1 - X^2](T)^{1/2} \\ &\leq 6\varepsilon\mathbb{E}\sup_{t\leq T}|w^1(t) - w^2(t)|^2 + 6\varepsilon^{-1}\mathbb{E}[X^1 - X^2](T) \\ &\leq 6\varepsilon\mathbb{E}\sup_{t\leq T} e^{-2\eta t}|u^1_{\lambda\beta}(t) - u^2_{\lambda\beta}(t)|^2 \\ &\quad + 6\varepsilon^{-1}\mathbb{E}\int_{Z_T}|G_1(s,z) - G_2(s,z)|^2\, m(dz)\, ds.\end{aligned}$$

Therefore we have

$$\begin{aligned}&(1 - 6\varepsilon)\mathbb{E}\sup_{t\leq T} e^{-2\eta t}|u^1_{\lambda\beta}(t) - u^2_{\lambda\beta}(t)|^2 \\ &\qquad \leq \mathbb{E}|x_1 - x_2|^2 + (1 + 6\varepsilon^{-1})\mathbb{E}\int_{Z_T}|G_1(s,z) - G_2(s,z)|^2\, m(dz)\, ds,\end{aligned}$$

hence, passing to the limit as β and λ go to zero, we obtain (14). □

Proposition 10. *Assume that $x \in \mathbb{L}_2$ and $G \in \mathbb{L}_2(L_2(Z_T))$. Then (2) admits a unique càdlàg generalized mild solution $u \in \mathbb{H}_2(T)$. Moreover, one has $x \mapsto u \in \dot{C}^{0,1}(\mathbb{L}_2, \mathbb{H}_2(T))$.*

Proof. Let us choose a sequence $\{x_n\} \subset \mathbb{L}_{2d}$ such that $x_n \to x$ in $\mathbb{L}_2$, and a sequence $\{G_n\} \subset \mathcal{L}_{d^*}$ such that $G_n \to G$ in $\mathbb{L}_2(L_2(Z_T))$ (e.g. by using a cut-off procedure). By Proposition 7 the stochastic equation

$$du + Au\, dt + f(u)\, dt = \eta u\, dt + G_n\, d\bar\mu, \qquad u(0) = x_n$$

admits a unique mild solution u_n. Then (14) yields

$$\mathbb{E}\sup_{t\leq T}|u_n(t)-u_m(t)|^2 \lesssim \mathbb{E}|x_n-x_m|^2+\mathbb{E}\int_{Z_T}|G_n(s,z)-G_m(s,z)|^2\,m(dz)\,ds$$

In particular $\{u_n\}$ is a Cauchy sequence in $\mathbb{H}_2(T)$, whose limit $u \in \mathbb{H}_2(T)$ is a generalized mild solution of (2). Since u_n is càdlàg for each n by Proposition 7, u is also càdlàg.

Moreover, it is immediate that $x_i \mapsto u_i$, $i=1,2$, satisfies $\|u_1-u_2\|_2^2 \lesssim |x_1-x_2|^2_{\mathbb{L}_2}$, i.e. the solution map is Lipschitz, which in turn implies uniqueness of the generalized mild solution. □

Remark 11. One could also prove well-posedness in $\mathcal{H}_2(T)$, simply using estimate (13) instead of (14). In this case one can also get explicit estimates for the Lipschitz constant of the solution map. On the other hand, one cannot conclude that a solution in $\mathcal{H}_2(T)$ is càdlàg, as the subset of càdlàg processes is not closed in $\mathcal{H}_2(T)$.

2.2. **Multiplicative noise.** Let us consider the stochastic evolution equation

$$du(t)+Au(t)\,dt+f(u(t))\,dt=\eta u(t)\,dt+\int_Z G(s,z,u(s-))\,\bar{\mu}(ds,dz) \tag{17}$$

with initial condition $u(0)=x$, where $G:\Omega\times[0,T]\times Z\times\mathbb{R}\times D\to\mathbb{R}$ is a $\mathcal{P}\otimes\mathcal{Z}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^n)$-measurable function, and we denote its associated Nemitski operator, which is a mapping from $\Omega\times[0,T]\times Z\times H\to H$, again by G.

We have the following well-posedness result for (17) in the generalized mild sense.

Theorem 12. *Assume that $x\in\mathbb{L}^2$ and G satisfies the Lipschitz condition*

$$\mathbb{E}\int_Z |G(s,z,u)-G(s,z,v)|^2\,m(dz)\,ds \leq h(s)|u-v|^2,$$

where $h\in L_1([0,T])$. Then (17) admits a unique generalized solution $u\in\mathbb{H}_2(T)$. Moreover, the solution map is Lipschitz from $\mathbb{L}_2$ to $\mathbb{H}_2(T)$.

Proof. For $v\in\mathbb{H}_2(T)$ and càdlàg, consider the equation

$$du(t)+Au(t)\,dt+f(u(t))\,dt=\eta u(t)\,dt+\int_Z G(s,z,v(s-))\,\bar{\mu}(ds,dz),\qquad u(0)=x. \tag{18}$$

Since $(s,z)\mapsto G(s,z,v(s-))$ satisfies the hypotheses of Proposition 10, (18) admits a unique generalized mild solution belonging to $\mathbb{H}_2(T)$. Let us denote the map associating v to u by F. We are going to prove that F is well-defined and is a contraction on $\mathbb{H}_{2,\alpha}(T)$ for a suitable choice of $\alpha>0$. Setting $u^i=F(v^i)$, $i=1,2$, with v^1, $v^2\in\mathbb{H}_2(T)$, we have

$$\begin{aligned} &d(u^1-u^2)+[A(u^1-u^2)+f(u^1)-f(u^2)]\,dt\\ &\qquad\qquad = \eta(u^1-u^2)\,dt+\int_Z[G(\cdot,\cdot,v^1_-)-G(\cdot,\cdot,v^2_-)]\,d\bar{\mu}\end{aligned}$$

in the mild sense, with obvious meaning of the (slightly simplified) notation. We are going to assume that u^1 and u^2 are strong solutions, without loss of generality: in fact, one otherwise approximate A, f and G with A_β, f_λ, and G_n, respectively, and passes to the limit in equation (19) below, leaving the rest of argument unchanged. Setting

$w^i(t) = e^{-\alpha t} u^i(t)$, $i = 1, 2$, we have, by an argument completely similar to the one used in the proof of Lemma 9,

$$\begin{aligned}|w^1(t) - w^2(t)|^2 &\leq (\eta - \alpha) \int_0^t e^{-2\alpha s} |u^1(s) - u^2(s)|^2 \, ds + [w^1 - w^2](t) \\ &\quad + 2 \int_{Z_t} \big\langle e^{-2\alpha s} (u^1(s-) - u^2(s-), (G(s, z, v^1(s-)) - G(s, z, v^2(s-)))\, \bar{\mu}(ds, dz) \big\rangle.\end{aligned}$$

The previous inequality in turn implies

$$\begin{aligned}\|u^1 - u^2\|_{2,\alpha}^2 \leq (\eta - \alpha) &\int_0^T \mathbb{E} \sup_{s \leq t} e^{-2\alpha s} |u^1(s) - u^2(s)|^2 \, ds \\ &+ 2\mathbb{E} \sup_{t \leq T} \big|(w^1_- - w^2_-) \cdot (X^1 - X^2)\big| \\ &+ \mathbb{E} \int_0^T \int_Z e^{-2\alpha s} |G(s, z, v^1(s-)) - G(s, z, v^2(s-))|^2 \, m(dz) \, ds,\end{aligned}$$

where we have set $X^i := G(\cdot, \cdot, v^i_-) \star \bar{\mu}$ and we have used the chain of identities

$$\begin{aligned}\mathbb{E} \sup_{t \leq T} [w^1 - w^2](t) &= \mathbb{E}[w^1 - w^2](T) \\ &= \mathbb{E} \int_0^T \int_Z e^{-2\alpha s} |G(s, z, v^1(s)) - G(s, z, v^2(s))|^2 \, m(dz) \, ds.\end{aligned}$$

An application of Davis' and Young's inequalities, as in the proof of Lemma 9, yields

$$\begin{aligned}2\mathbb{E} \sup_{t \leq T} \big|(w^1_- - w^2_-) \cdot (X^1 - X^2)\big| &\leq 6\varepsilon \mathbb{E} \sup_{t \leq T} |w^1(t) - w^2(t)|^2 \\ &\quad + 6\varepsilon^{-1} \mathbb{E} \int_0^T \int_Z e^{-2\alpha s} |G(s, z, v^1(s)) - G(s, z, v^2(s))|^2 \, m(dz) \, ds,\end{aligned}$$

because $[X^1 - X^2] = [w^1 - w^2]$. We have thus arrived at the estimate

$$\begin{aligned}(1 - 6\varepsilon) \|u^1 - u^2\|_{2,\alpha}^2 &\leq (\eta - \alpha) \int_0^T \mathbb{E} \sup_{s \leq t} e^{-2\alpha s} |u^1(s) - u^2(s)|^2 \, dt \\ &\quad + (1 + 6\varepsilon^{-1}) \mathbb{E} \int_0^T \int_Z e^{-2\alpha s} |G(s, z, v^1(s)) - G(s, z, v^2(s))|^2 \, m(dz) \, ds \qquad (19)\end{aligned}$$

Setting $\varepsilon = 1/12$ and $\phi(t) = \mathbb{E} \sup_{s \leq t} e^{-2\alpha s} |u^1(s) - u^2(s)|^2$, we can write, by the hypothesis on G,

$$\phi(T) \leq 2(\eta - \alpha) \int_0^T \phi(t) \, dt + 146 |h|_{L_1} \|v^1 - v^2\|_{2,\alpha}^2,$$

hence, by Gronwall's inequality,

$$\|u^1 - u^2\|_{2,\alpha}^2 = \phi(T) \leq 146 |h|_1 e^{2(\eta - \alpha)T} \|v^1 - v^2\|_{2,\alpha}^2.$$

Choosing α large enough, we obtain that there exists a constant $N = N(T) < 1$ such that $\|F(v^1) - F(v^2)\|_{2,\alpha} \leq N \|v^1 - v^2\|_{2,\alpha}$. Banach's fixed point theorem then implies that F admits a unique fixed point in $\mathbb{H}_{2,\alpha}(T)$, which is the (unique) generalized solution of (17), recalling that the norms $\|\cdot\|_{2,\alpha}$, $\alpha \geq 0$, are all equivalent. Since the fixed point

of F can also be obtained as a limit of càdlàg processes in $\mathbb{H}_2(T)$, by the well-known method of Picard's iterations, we also infer that the generalized mild solution is càdlàg.

Moreover, denoting $u(x_1)$ and $u(x_2)$ by u^1 and u^2 respectively, an argument similar to the one leading to (19) yields the estimate

$$\psi(T) \leq \mathbb{E}|x_1 - x_2|^2 + 2(\eta - \alpha) \int_0^T \psi(t)\, dt + 146 \int_0^T h(t)\psi(t)\, dt,$$

where $\psi(t) := \mathbb{E} \sup_{s \leq t} |u^1(s) - u^2(s)|^2$. By Gronwall's inequality we get

$$\|u^1 - u^2\|_{2,\alpha}^2 \leq e^{2(\eta-\alpha)+146|h|_{L_1}} |x_1 - x_2|_{\mathbb{L}_2}^2,$$

which proves that $x \mapsto u(x)$ is Lipschitz from $\mathbb{L}_2$ to $\mathbb{H}_{2,\alpha}(T)$, hence also from $\mathbb{L}_2$ to $\mathbb{H}_2(T)$ by the equivalence of the norms $\|\cdot\|_{2,\alpha}$. □

3. Invariant measures and Ergodicity

Throughout this section we shall assume that $G : Z \times H \to H$ is a (deterministic) $\mathcal{Z} \otimes \mathcal{B}(H)$-measurable function satisfying the Lipschitz assumption

$$\int_Z |G(z,u) - G(z,v)|^2\, m(dz) \leq K|u - v|^2,$$

for some $K > 0$. The latter assumption guarantees that the evolution equation is well-posed by Theorem 12. Moreover, it is easy to see that the solution is Markovian, hence it generates a semigroup via the usual formula $P_t\varphi(x) := \mathbb{E}\varphi(u(t,x))$, $\varphi \in B_b(H)$. Here $B_b(H)$ stands for the set of bounded Borel function from H to $\mathbb{R}$.

3.1. **Strongly dissipative case.**

Throughout this subsection we shall assume that there exist β_0 and $\omega_1 > 0$ such that

$$2\langle A_\beta u - A_\beta v, u - v\rangle + 2\langle f_\lambda(u) - f_\lambda(v), u - v\rangle - 2\eta|u - v|^2 - \int_Z |G_n(z,u) - G_n(z,v)|^2\, m(dz) \geq \omega_1 |u - v|^2 \tag{20}$$

for all $\beta \in]0, \beta_0[$, $\lambda \in]0, \beta_0[$, $n > [1/\beta_0]$, and for all u, $v \in H$, where $[x]$ stands for the integer part of $x \in \mathbb{R}$. This is enough to guarantee existence and uniqueness of an ergodic invariant measure for P_t, with exponentially fast convergence to equilibrium.

Proposition 13. *Under hypothesis (20) there exists a unique invariant measure ν for P_t, which satisfies the following properties:*

(i) $\displaystyle\int |x|^2\, \nu(dx) < \infty$;

(ii) *let $\varphi \in \dot{C}^{0,1}(H,\mathbb{R})$ and $\lambda_0 \in \mathcal{M}_1(H)$. Then one has*

$$\Big| P_t\varphi(x)\, \lambda_0(dx) - \int_H \varphi\, \nu(dy) \Big| \leq [\varphi]_1 e^{-\omega_1 t} \int_{H\times H} |x - y|\, \lambda_0(dx)\, \nu(dy)$$

Following a classical procedure (see e.g. [13, 30, 31]), let us consider the stochastic equation

$$du(t) + (Au(t) + f(u))\, dt = \eta u(t)\, dt + \int_Z G(z, u(t-))\, d\bar{\mu}_1(dt,dz), \qquad u(s) = x, \tag{21}$$

where $s \in]-\infty, t[$, $\bar{\mu}_1 = \mu_1 - \mathrm{Leb} \otimes m$, and

$$\mu_1(t,B) = \begin{cases} \mu(t,B), & t \geq 0, \\ \mu_0(-t,B), & t < 0, \end{cases}$$

for all $B \in \mathcal{Z}$, with μ_0 an independent copy of μ. The filtration on which μ_1 is considered can be constructed along the lines of [31, p. 99], using in their argument the Poisson point processes associated to μ and μ_0 instead of Wiener processes. We shall denote the value at time $t \geq s$ of the solution of (21) by $u(t; s, x)$.

For the proof of Proposition 13 we need the following lemma.

Lemma 14. *There exists a random variable $\zeta \in \mathbb{L}_2$ such that $u(0; s, x) \to \zeta$ in $\mathbb{L}_2$ as $s \to -\infty$ for all $x \in \mathbb{L}_2$. Moreover, there exists a constant N such that*

$$\mathbb{E}|u(0; s, x) - x|^2 \leq e^{-2\omega_1 |s|} N(1 + \mathbb{E}|x|^2) \tag{22}$$

for all $s < 0$.

Proof. Let us denote the strong solution of the approximating equation by $u^n_{\lambda\beta}$. By Itô's lemma we can write

$$\begin{aligned} &|u^n_{\lambda\beta}(t)|^2 + 2\int_s^t \left[\langle A_\beta u^n_{\lambda\beta}(r), u^n_{\lambda\beta}(r)\rangle + \langle f_\lambda(u^n_{\lambda\beta}(r)), u^n_{\lambda\beta}(r)\rangle - \eta |u^n_{\lambda\beta}(r)|^2\right] dr \\ &= |x|^2 + 2\int_s^t\int_Z \langle G_n(z, u^n_{\lambda\beta}(r-)), u^n_{\lambda\beta}(r)\rangle\, \bar{\mu}(dr,dz) + \int_s^t\int_Z |G_n(z, u^n_{\lambda\beta}(r-))|^2\, \mu(dr,dz). \end{aligned} \tag{23}$$

Note that we have, by Young's inequality,

$$\begin{aligned} -\langle f_\lambda(u), u\rangle &= -\langle f_\lambda(u) - f(0), u - 0\rangle - \langle f_\lambda(0), u\rangle \\ &\leq -\langle f_\lambda(u) - f(0), u - 0\rangle + \frac{\varepsilon}{2}|u|^2 + \frac{1}{2\varepsilon}|f_\lambda(0)|^2 \end{aligned}$$

and, similarly,

$$\begin{aligned} \int_Z |G_n(u,z)|^2\, m(dz) &= \int_Z |G_n(u,z) - G_n(0,z) + G_n(0,z)|^2\, m(dz) \\ &\leq \int_Z |G_n(u,z) - G_n(0,z)|^2\, m(dz) + \varepsilon K|u|^2 + (1+\varepsilon^{-1})\int_Z |G_n(0,z)|^2\, m(dz). \end{aligned}$$

Since $f_\lambda(0) \to f(0)$ as $\lambda \to 0$ and $G_n(0,\cdot) - G(0,\cdot) \to 0$ in $L_2(Z,m)$ as $n \to \infty$, there exists $\delta > 0$, $\lambda_0 > 0$ and $n_0 > 0$ such that

$$|f_\lambda(0)|^2 \leq |f(0)|^2 + \delta/2 \qquad \forall \lambda < \lambda_0,$$

$$(1+\varepsilon)^{-1}\int_Z |G_n(0,z)|^2\, m(dz) \leq (1+\varepsilon)^{-1}\int_Z |G(0,z)|^2\, m(dz) + \delta/2 \qquad \forall n > n_0.$$

Therefore we have, for $\beta < \beta_0$, $\lambda < \lambda_0 \wedge \beta_0$, and $n > n_0 \vee [1/\beta_0]$,

$$\begin{aligned} &-2\langle A_\beta u^n_{\lambda\beta}, u^n_{\lambda\beta}\rangle - 2\langle f_\lambda(u^n_{\lambda\beta}), u^n_{\lambda\beta}\rangle + 2\eta|u^n_{\lambda\beta}|^2 + \int_Z |G_n(u^n_{\lambda\beta}, z)|^2\, m(dz) \\ &\quad \leq -\omega_1 |u^n_{\lambda\beta}|^2 + \varepsilon(1+K)|u^n_{\lambda\beta}|^2 + \varepsilon^{-1}|f(0)|^2 + (1+\varepsilon^{-1})\int_Z |G(0,z)|^2\, m(dz) + \delta. \end{aligned}$$

Taking expectations in (23), applying (20), and passing t the limit as $\beta \to 0$, $\lambda \to 0$, and $n \to \infty$, we are left with

$$\mathbb{E}|u(t)|^2 \leq \mathbb{E}|x|^2 - \omega_2 \int_s^t \mathbb{E}|u(r)|^2 \, dr + N,$$

where $\omega_2 := \omega_1 - (1+K)\varepsilon$ and $N := \varepsilon^{-1}(|f(0)|^2 + (1+\varepsilon^{-1})|G(0,\cdot)|^2_{L_2(Z,m)}) + \delta$. We choose ε so that $\omega_2 > 0$. Gronwall's inequality then yields

$$\mathbb{E}|u(t)|^2 \lesssim 1 + e^{-2\omega_2(t+|s|)}\mathbb{E}|x|^2. \tag{24}$$

Set $u_1(t) := u(t; s_1, x)$, $u_2(t) := u(t; s_2, x)$ and $w(t) = u_1(t) - u_2(t)$, with $s_2 < s_1$. Then w satisfies the equation

$$dw + Aw \, dt + (f(u_1) - f(u_2)) \, dt = \eta w \, dt + (G(u_1) - G(u_2)) \, d\bar{\mu},$$

with initial condition $w(s_1) = x - u_2(s_1)$, in the generalized mild sense. By an argument completely similar to the above one, based on regularizations, Itô's formula, and passage to the limit, we obtain, thanks to (20),

$$\mathbb{E}|w(t)|^2 \leq \mathbb{E}|x - u_2(s_1)|^2 - \omega_1 \int_{s_1}^t \mathbb{E}|w(r)|^2 \, dr,$$

and hence, by Gronwall's inequality,

$$\mathbb{E}|u_1(0) - u_2(0)|^2 = \mathbb{E}|w(0)|^2 \leq e^{-2\omega_1|s_1|}\mathbb{E}|x - u_2(s_1)|^2.$$

Estimate (24) therefore implies that there exists a constant N such that

$$\mathbb{E}|u_1(0) - u_2(0)|^2 \leq e^{-2\omega_1|s_1|}N(1 + \mathbb{E}|x|^2), \tag{25}$$

which converges to zero as $s_1 \to -\infty$. We have thus proved that $\{u(0; s, x)\}_{s \leq 0}$ is a Cauchy net in $\mathbb{L}_2$, hence there exists $\zeta = \zeta(x) \in \mathbb{L}_2$ such that $u(0; s, x) \to \zeta$ in $\mathbb{L}_2$ as $s \to -\infty$. Let us show that ζ does not depend on x. In fact, let x, $y \in \mathbb{L}_2$ and set $u_1(t) = u(t; s, x)$, $u_2(t) = u(t; s, y)$. An argument based on Itô's formula for the square of the norm and the monotonicity assumption (20) yields, in analogy to a previous computation,

$$\mathbb{E}|u_1(0) - u_2(0)|^2 \leq e^{-2\omega_1|s|}\mathbb{E}|x - y|^2, \tag{26}$$

which implies $\zeta(x) = \zeta(y)$, whence the claim. Finally, (25) immediately yields (22). □

Proof of Proposition 13. Let ν be the law of the random variable ζ constructed in the previous lemma. Since $\zeta \in \mathbb{L}_2$, (i) will follow immediately once we have proved that ν is invariant for P_t. The invariance and the uniqueness of ν is a well-known consequence of the previous lemma, see e.g. [10].

Let us prove (ii): we have

$$\begin{aligned}
&\Big| \int_H P_t\varphi(x)\,\lambda_0(dx) - \int_H \varphi(y)\,\nu(dy)\Big| \\
&\qquad = \Big| \int_H\int_H P_t\varphi(x)\,\lambda_0(dx)\,\nu(dy) - \int_H\int_H P_t\varphi(y)\,\lambda_0(dx)\,\nu(dy)\Big| \\
&\qquad \leq \int_{H\times H} |P_t\varphi(x) - P_t\varphi(y)|\,\lambda_0(dx)\,\nu(dy) \\
&\qquad \leq [\varphi]_1 e^{-\omega_1 t} \int_{H\times H} |x-y|\,\lambda_0(dx)\,\nu(dy),
\end{aligned}$$

where in the last step we have the estimate (26). □

3.2. **Weakly dissipative case.** In this subsection we replace the strong dissipativity condition (20) with a super-linearity assumption on the nonlinearity f, and we prove existence of an invariant measure by an argument based on Krylov-Bogoliubov's theorem.

We assume that $-A$ satisfies the weak sector condition and let $(\mathcal{E}, D(\mathcal{E}))$ be the associated closed coercive form (see [26, §I.2]). We set $\mathcal{H} := D(\mathcal{E})$, endowed with the norm associated to the inner product $\mathcal{E}_1(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + \langle\cdot,\cdot\rangle$.

Theorem 15. *Assume that*

(i) *f satisfies the super-linearity condition $\langle f(r), r\rangle \geq b|r|^{2(1+\alpha)}$, $b > 0$, $\alpha > 0$.*
(ii) *$\mathcal{H}$ is compactly embedded into $L_2(D)$.*

Then there exists an invariant measure for the transition semigroup associated to the generalized mild solution of (17).

Proof. Let $u^n_{\lambda\beta}$ denote the strong solution of the equation obtained by replacing in (17) A with A_β, f with f_λ, and G with G_n. Then an application of Itô's formula yields the estimate

$$\begin{aligned}
\mathbb{E}|u^n_{\lambda\beta}(t)|^2 &+ 2\mathbb{E}\int_0^t \big[\langle A_\beta u^n_{\lambda\beta}(s), u^n_{\lambda\beta}(s)\rangle + \langle f_\lambda(u^n_{\lambda\beta}(s)), u^n_{\lambda\beta}(s)\rangle\big]\,ds \\
&\leq \mathbb{E}|x|^2 + \mathbb{E}\int_0^t \big[2\eta|u^n_{\lambda\beta}(s)|^2 + |G_n(u^n_{\lambda\beta}(s))|^2_{L_2(Z,m)}\big]\,ds. \quad (27)
\end{aligned}$$

Since

$$\langle f_\lambda(r), r\rangle = \langle f(J_\lambda r), J_\lambda r + (r - J_\lambda r)\rangle = \langle f(J_\lambda r), J_\lambda r\rangle + \lambda|f_\lambda(r)|^2,$$

we obtain, taking into account the monotonicity of A_β and the uniform (in n) Lipschitz continuity of G_n,

$$\mathbb{E}|u^n_{\lambda\beta}(t)|^2 \leq \mathbb{E}|x|^2 + N\big(1 + \int_0^t \mathbb{E}|u^n_{\lambda\beta}(s)|^2\,ds\big) - \mathbb{E}\int_0^t \langle f(J_\lambda u^n_{\lambda\beta}(s)), J_\lambda u^n_{\lambda\beta}(s)\rangle\,ds$$

for some constant $N>0$ that does not depend on λ, β, and n. By assumption (i) and Jensen's inequality, we have

$$\begin{aligned}-\int_0^t \mathbb{E}\langle f(J_\lambda u^n_{\lambda\beta}(s)), J_\lambda u^n_{\lambda\beta}(s)\rangle\,ds &\leq -b\int_0^t \mathbb{E}|J_\lambda u^n_{\lambda\beta}(s)|^{2+2\alpha}\,ds\\ &\leq -b\int_0^t \big(\mathbb{E}|J_\lambda u^n_{\lambda\beta}(s)|^2\big)^{1+\alpha}\,ds,\end{aligned}$$

thus also

$$\mathbb{E}|u^n_{\lambda\beta}(t)|^2 \leq \mathbb{E}|x|^2 + N\big(1+\int_0^t \mathbb{E}|u^n_{\lambda\beta}(s)|^2\,ds\big) - b\int_0^t \big(\mathbb{E}|J_\lambda u^n_{\lambda\beta}(s)|^2\big)^{1+\alpha}\,ds.$$

Recalling that

$$u^n_{\lambda\beta} \xrightarrow{\beta\to 0} u^n_\lambda \xrightarrow{\lambda\to 0} u^n \xrightarrow{n\to\infty} u$$

strongly in $\mathbb{H}_2(T)$, passing to the limit in the previous inequality shows that $y(t) := \mathbb{E}|u(t)|^2$ satisfies the differential inequality (in its integral formulation, to be more precise)

$$y' \leq Ny - by^{1+\alpha}.$$

By simple ODE techniques one obtains that $y(t)$ is bounded for all t, i.e. $\mathbb{E}|u(t)|^2 \leq C$ for all $t\geq 0$, for some positive constant C.

Taking into account the monotonicity of f_λ and the uniform Lipschitz continuity of G_n, (27) also implies

$$\begin{aligned}&\mathbb{E}\int_0^t \langle A_\beta u^n_{\lambda\beta}(s), u^n_{\lambda\beta}(s)\rangle\,ds\\ &\qquad\leq \mathbb{E}|x|^2 + N\big(1+\int_0^t \mathbb{E}|u^n_{\lambda\beta}(s)|^2\,ds\big) - b\int_0^t \big(\mathbb{E}|J_\lambda u^n_{\lambda\beta}(s)|^2\big)^{1+\alpha}\,ds, \quad (28)\end{aligned}$$

for a constant N independent of λ, β and n. As we have seen above, the right-hand side of the inequality converges, as $\beta\to 0$, $\lambda\to 0$, $n\to\infty$, to

$$\mathbb{E}|x|^2 + N\big(1+\int_0^t \mathbb{E}|u(s)|^2\,ds\big) - b\int_0^t \big(\mathbb{E}|u(s)|^2\big)^{1+\alpha}\,ds \lesssim 1+t.$$

In analogy to an earlier argument, setting $z_\beta := (I+\beta A)^{-1}z$, $z\in H$, we have

$$\langle A_\beta z, z\rangle = \langle Az_\beta, z_\beta + z - z_\beta\rangle = \langle Az_\beta, z_\beta\rangle + \beta|A_\beta z|^2.$$

In particular, setting $v^n_{\lambda\beta} := (I+\beta A)^{-1}u^n_{\lambda\beta} \in D(\mathcal{E})$, we obtain

$$\begin{aligned}\mathbb{E}\int_0^t \mathcal{E}\big(v^n_{\lambda\beta}(s), v^n_{\lambda\beta}(s)\big)\,ds &\leq \mathbb{E}\int_0^t \langle A_\beta u^n_{\lambda\beta}(s), u^n_{\lambda\beta}(s)\rangle\,ds\\ &\leq \mathbb{E}|x|^2 + N\big(1+\int_0^t \mathbb{E}|u^n_{\lambda\beta}(s)|^2\,ds\big) - b\int_0^t \big(\mathbb{E}|J_\lambda u^n_{\lambda\beta}(s)|^2\big)^{1+\alpha}\,ds,\end{aligned}$$

Since $u^n_{\lambda\beta}\to u$ in $\mathbb{H}_2(T)$ for all $T\geq 0$ and $\mathbb{E}|u(t)|^2 \lesssim 1$, we also get

$$\mathbb{E}\int_0^t \mathcal{E}\big(v^n_{\lambda\beta}(s), v^n_{\lambda\beta}(s)\big)\,ds \lesssim 1+t.$$

Since also $v^n_{\lambda\beta}(s) \to u(s)$ in $\mathbb{H}_2(T)$, it follows that $u \in L_2(\Omega \times [0,t], D(\mathcal{E}))$ and $v^n_{\lambda\beta} \to u$ weakly in $L_2(\Omega \times [0,t], D(\mathcal{E}))$, where $D(\mathcal{E})$ is equipped with the norm $\mathcal{E}_1^{1/2}(\cdot,\cdot)$, and

$$\mathbb{E}\int_0^t \mathcal{E}_1(u(s),u(s))\,ds \lesssim 1+t.$$

Let us now define the sequence of probability measures $(\nu_n)_{n\geq 1}$ on the Borel set of $H = L_2(D)$ by

$$\int_H \phi\,d\nu_n = \frac{1}{n}\int_0^n \mathbb{E}\phi(u(s,0))\,ds, \qquad \phi \in B_b(H).$$

Then

$$\int |x|^2_{\mathcal{H}}\,\nu_n(dx) = \frac{1}{n}\int_0^n \mathbb{E}\,\mathcal{E}_1(u(s,0),u(s,0)\,ds \lesssim 1,$$

thus also, by Markov's inequality,

$$\sup_{n\geq 1}\nu_n(B_R^c) \lesssim \frac{1}{R} \xrightarrow{R\to\infty} 0,$$

where B_R^c stands for the complement in $\mathcal{H}$ of the closed ball of radius R in $\mathcal{H}$. Since balls in $\mathcal{H}$ are compact sets of $L_2(D)$, we infer that $(\nu_n)_{n\geq 1}$ is tight, and Krylov-Bogoliubov's theorem guarantees the existence of an invariant measure. □

Appendix A. Auxiliary results

The following proposition is a slight modification of [29, Thm. 6.1.2] and it is used in the proof of Lemma 6. Here $[0,T] \subset \mathbb{R}$ and E is a separable Banach space.

Proposition 16. *Assume that $f : [0,T] \times E \to E$ satisfies*

$$|f(t,x) - f(t,y)| \leq N|x-y|, \qquad \forall t \in [0,T],\ x,\ y \in E,$$

where N is a constant independent of t, and there exists $a \in E$ such that $t \mapsto f(t,a) \in L_1([0,T];E)$. If A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on E and $u_0 \in E$, then the integral equation

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s,u(s))\,ds, \qquad t \in [0,T], \tag{29}$$

admits a unique solution $u \in C([0,T],E)$.

Proof. As a first step, let us show that, if $v \in L_\infty([0,T];E)$, then $t \mapsto f(t,v(t)) \in L_1([0,T];E)$. In fact, we have

$$\begin{aligned}|f(t,v(t))| &\leq |f(t,v(t)) - f(t,a)| + |f(t,a)|\\ &\leq N|v(t)-a| + |f(t,a)| \leq N|v(t)| + N|a| + |f(t,a)|,\end{aligned}$$

thus also

$$\int_0^T |f(t,v(t))|\,dt \leq NT|a| + NT|v|_{L_\infty} + |f(\cdot,a)|_{L_1} < \infty.$$

As a second step, we show that the map

$$[\mathfrak{F}v](t) := e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s,v(s))\,ds$$

is a (local) contraction in $L_\infty([0,T];E)$. In fact, setting $M_T = \sup_{t\in[0,T]} |e^{tA}|$, we have

$$\sup_{t\in[0,T]} \big|[\mathfrak{F}(v)](t)\big| \le M_T|u_0| + M_T \int_0^T |f(s,v(s)|\,ds < \infty,$$

because $f(\cdot,v(\cdot)) \in L_1([0,T];E)$, as proved above. We also have

$$\begin{aligned}\sup_{t\in[0,T]} \big|[\mathfrak{F}v](t) - [\mathfrak{F}w](t)\big| &\le NM_T \sup_{t\in[0,T]} \int_0^t |v(s)-w(s)|\,ds \\ &\le NM_T T|v-w|_{L_\infty},\end{aligned}$$

so that $NM_TT_0 < 1$ for T_0 small enough. Then $\mathfrak{F}$ admits a unique fixed point in $L_\infty([0,T_0];E)$, and by a classical patching argument we obtain the existence of a unique solution $u \in L_\infty([0,T];E)$ to the integral equation (29). As a last step, it remains to prove that $u \in C([0,T];E)$. To this purpose, it suffices to show that $g \in L_1([0,T];E)$ implies $F \in C([0,T];E)$, with

$$F(t) := \int_0^t e^{(t-s)A} g(s)\,ds.$$

In fact, for $0 \le t < t+\varepsilon < T$, we have

$$\begin{aligned}|F(t+\varepsilon)-F(t)| &\le \Big|\int_0^t [e^{(t+\varepsilon-s)A}g(s) - e^{(t-s)A}g(s)]\,ds\Big| + \Big|\int_t^{t+\varepsilon} e^{(t+\varepsilon-s)A}g(s)\,ds\Big| \\ &\le |e^{\varepsilon A} - I|M_T \int_0^t |g(s)|\,ds + M_T\int_t^{t+\varepsilon} |g(s)|\,ds,\end{aligned}$$

and both terms converge to zero as $\varepsilon \to 0$ by definition of strongly continuous semigroup and because $g \in L_1([0,T],E)$. The case $0 < t-\varepsilon < t \le T$ is completely similar, hence omitted. □

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